1 Small Ball Mean Value Property

Small ball mean value property (SBMVP) means that for every $x \in \Omega$, there is a $r(x)$ such that f satisfies MVP (mean value property) for $B_r(x)$, where $r \le r(x)$, that is,

$$\fint_{B_r(x)} f = f(x)$$

for $r \le r(x)$. We know that MVP implies harmonicity. This fact could also be proved by the solvability of the Dirichlet problem for balls. Moreover, we could obtain a stronger proposition that improves MVP to SBMVP.

Exercise 1.1
If a continuous function u satisfies SBMVP on Ω, then u is harmonic on Ω.

Proof. We first use SBMVP to establish MP (maximum principle). Assume that Ω is connected. Suppose that $u \in C(\overline{\Omega})$ and u satisfies SBMVP on Ω. We aim to prove that

$$\max_{\overline{\Omega}} u = \max_{\partial\Omega} u.$$

Let $M = \max_{\overline{\Omega}} u$, then we claim that $\{x \in \Omega \mid u(x) = M\}$ is closed and open. Clearly, it is closed. Next we show that it is open. Suppose that $u(x_0) = M$, then by SBMVP, $u = M$ around x_0. So it is open. Now if the maximum is attained in Ω, then $u = M$ on Ω, so it is also attained on boundary. So MP is proved.

Next we show that u is harmonic. For $x_0 \in \Omega$, we show that u is harmonic on a ball B around x_0. Suppose that

$$\begin{aligned} \Delta\widetilde{u} &= 0 \quad \text{in } B; \\ \widetilde{u} &= u \quad \text{on } \partial B. \end{aligned}$$

Then $\widetilde{u}$ is harmonic in B. If we show that $\widetilde{u} = u$ in B, the proof is concluded. Since $\widetilde{u} - u$ satisfies SBMVP on B and vanishes on ∂B, by MP, $\widetilde{u} = u$ in B. $\square$

As an application of the above proposition, we use it prove the Schwarz reflection principle.

Theorem 1.1 (Schwarz Reflection Principle)
Suppose that Ω is a region in $\mathbb{R}^{n+1}$ that is symmetric with respect to $x_{n+1} = 0$.

Let $\Omega_+ = \Omega \cap \{x_{n+1} > 0\}$*,* $\Gamma = \Omega \cap \{x_{n+1} = 0\}$*, and* $\Omega_- = \Omega \cap \{x_{n+1} < 0\}$*. Suppose that* u *is harmonic in* Ω_+ *and is continuous on* $\Omega_+ \cup \Gamma$*. Moreover,* $u = 0$ *on* Γ*. Then* u *can be extended to a harmonic function defined on* Ω*. In fact, the extended function is*

$$\widetilde{u} = \begin{cases} u & \textit{on } \Omega_+; \\ 0 & \textit{on } \Gamma; \\ -u & \textit{on } \Omega_-. \end{cases}$$

To prove it, it is easy to verify that $\widetilde{u}$ satisfies SBMVP on Ω. Moreover, it we use MVP to show the harmonicity, it is somewhat difficult to verify the balls intersects both Ω_+ and Ω_-.

2 Positive Harmonic Functions

In this section, we prove a Liouville theorem for positive harmonic functions.

Exercise 2.1

If u is a positive harmonic function on $\mathbb{R}^n$, then u is a constant.

Remark 2.1

The complex analysis version of this proposition is that an entire function (holomorphic functions defined on $\mathbb{C}$) whose real part is positive is a constant. It is related to the little Picard theorem, which says that if an entire function omits two points (which means that there are two points lying outside the range of the function), then the function is a constant.

Proof. For x, y in $\mathbb{R}^n$, we aim to show that $u(x) = u(y)$. We use MVP to prove it. For every $R > 0$,

$$u(x) = \fint_{B_R(x)} u.$$

Since $B_R(x) \subset B_{R+|x-y|}(y)$,

$$\begin{aligned} u(x) &\leq \frac{1}{|B_R(x)|} \int_{B_{R+|x-y|}(y)} u \\ &= \frac{|B_{R+|x-y|}(y)|}{|B_R(x)|} u(y) \\ &= \left(\frac{R + |x-y|}{R} \right)^n u(y). \end{aligned}$$

Let $R \to \infty$, then it follows that

$$u(x) \leq u(y).$$

Similarly, $u(y) \leq u(x)$, and the proof is concluded. $\square$